THE ANALYSIS OF CYCLIC POINT PROCESSES WITH WALSH FUNCTIONS*

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Transformation by Walsh functions is a technique useful for uncovering the interactions among the daily, weekly, and seasonal cycles often present in societal data. The structural and sampling properties needed for interpretation are simple, the former because Walsh functions are easy to visualize, the latter because the normal approximation applies whenever the total number of events is large. This paper demonstrates these properties including their extension multivariate point processes. This technique is illustrated by applying it to a series of robberies.

1. INTRODUCTION

For many series of events, the rate of occurrence is modulated by periodic phenomena of known frequency and also non-periodic phenomena. Robberies are one illustration. Although the frequencies are known, the effects on the rate of occurrence of the various phenomena are neither additive nor multiplicative but unknown. In other words, the rate function changes from period to period under influences of longer-period phenomena and non-periodic phenomena, influences that interact in an unknown way. For example, the weekday rate function for robberies differs from the weekend rate function, the winter from the summer. That the rate of occurrence is small also motivates the analysis to be discussed.

The data being considered can be regarded as a point process with interesting non-stationary characteristics. Lewis [8] illustrates point-process methods with admissions to a hospital emergency room. Some point process methods [3, 7, 8] are appropriate, but others [2, 4], meant for stationary point processes, are not. The data can also be regarded as a sparse contingency table with interesting interactions. For example, the table might have as dimensions an appropriate division of the day, day of the week, and week of the year. Thus, contingency table methods [1, 6] are appropriate.

Properly applied, transformation by Walsh functions shows the periodicities, the interactions, and how rapidly the rate function changes within periods. As discussed below, the Walsh transform must be matched to the frequencies present. Some Walsh coefficients indicate periodicities because they are averages over all periods of a function of the counts within a period. Other Walsh coefficients indicate interactions because they are differences between periods of a function of the counts within a period. Further, Walsh coefficients indicate behavior of the rate function within periods because they are based on successively finer divisions of the time scale.

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Thus, the Walsh transform gives at once coefficients that contraindicate smoothing by combining cells (using a courser division of the interval) and coefficients that contraindicate smoothing by ignoring interactions. In the analysis of sparse tables, comparison of these types of smoothing is important.

One further type of smoothing should be considered, combining events that are somewhat different. Extension of the Walsh-transform technique to this case is indicated.

Because Walsh functions have a range of ± 1, each Walsh coefficient is the difference between numbers of events in a two-way partition of the total interval. Consequently, the easiest interpretation of the Walsh coefficients is in terms of an additive model. When compared to multiplicative models, additive models have disadvantages [1]. However, because the coefficients are differences of counts, the normal approximation applies regardless of the cell size. Thus, the distribution theory for Walsh coefficients is simple.

The proper application of the Walsh transform to most data requires adding to the series periods that contain no events. For example, an eighth day must be added to the week and twelve weeks must be added to the year. These additions are needed because the Walsh transform only matches frequencies that are a power of two times some lowest frequency. Because of these additions, some of the Walsh coefficients must be adjusted so they can be compared to zero.

The Walsh transform is like analysis of variance with orthogonal contrasts. These techniques are identical if the ratios of the cycle lengths are a power of two, as required for proper application of the Walsh transform. Alternatively, orthogonal contrasts could have been used without adding zeros. This latter approach does not give uncorrelated coefficients as is usually the case in analysis of variance because the process is nonhomogeneous. Further, the results of the latter approach are not as easy to interpret because the resulting coefficients are not simple differences of counts.

The Walsh transform has attracted attention in signal processing because it is easily computed, requiring only $n \log_2 n$ additions for an n-point transform [9, 10, 11].

2. A SERIES OF ROBBERIES

The data that motivated this paper are descriptions of robberies (the taking of something from a person by force or threat of force) and purse snatchings that occurred in the Bronx during 1969 and 1970. These descriptions were recorded on a special form by the New York City Police. The recording was ordered and carefully supervised by the commander of the Bronx, who used the data for operational purposes. Beside time and place of occurrence, these descriptions include characteristics of the victims, perpetrators, and circumstances of the crime. Many series of events recorded by institutions have similar statistical properties.

For these data, the shapes of the daily, weekly, and seasonal patterns, as well as how they vary with event characteristics, are of interest. However, the intent of this paper is not to present a picture of robbery but only to illustrate the Walsh transform. The incidents used for illustration are single-victim, non-commercial incidents in which the victims were males between the ages of 18 and 52. The interval examined is the 52 week period from Sunday, July 27, 1969 to Saturday, July 25, 1970. This interval is chosen so that the first and fourth quarters are in daylight saving time and the second and third quarters are in standard time.

In this application of the Walsh transform, the day is divided into eight subintervals: 1:01 a.m. to 9:00 a.m., 9:01 a.m. to 1:00 p.m., 1:01 p.m. to 3:15 p.m., 3:16 p.m. to 5:20 p.m., 5:21 p.m. to 6:45 p.m., 6:46 p.m. to 8:30 p.m., 8:31 p.m. to 10:15 p.m., and 10:16 p.m. to 1:00 a.m. The unequal subintervals are justified because the shortest period expected in robbery data is one day. Thus, the time scale for days can be replaced by any monotone function of time. The eight subintervals are chosen to equalize the number of single-victim non-commercial incidents in each subinterval. Because the data contain ties due to rounding of event times, this cannot be done exactly. Since the equalization is for victims of all sexes and ages, equalization for a particular victim sex-age category such as adult males is not expected.

To meet the requirement that the ratios of the periods must be a power of two, a day with zero events is added to each week and three weeks with zero events are added to each quarter. The zero-event day follows Sunday, becoming the second day of the week. The three zero-event weeks follow the last week of the quarter. Including the divisions of the day, this produces 4096 (2^{12}) subintervals. The Walsh transform is applied to the counts in these subintervals.

Because of the added zero-event sub-intervals, some comparisons computed by the Walsh transform are between subintervals of unequal length. The Walsh coefficients that compare weeks within a quarter compare either eight weeks to five weeks or seven weeks to six weeks. Thus, they will not be zero when the weeks of the quarter are the same. Similarly, the Walsh coefficients that compare days within a week compare four days to three days. Thus, they will not be zero when the seven days of the week are all the same. For example, the Walsh coefficient that is the difference between the number of events on Sunday, Friday, and Saturday and the number on Monday, Tuesday, Wednesday, and Thursday does not account for the unequal number of days in the two groups.

The proper adjustment is normalization by the lengths of the subintervals compared. Walsh coefficients can be obtained in four steps: by evaluating for each day a function of the occurrence times in that day, by evaluating for each week a function of the daily values obtained for that week, by evaluating for each quarter a function of the weekly values obtained for that quarter, and by evaluating for the year a function of the quarterly values obtained.

Each of these functions is either a sum of the values for the next shortest period or a difference of two sums of the values. Adjustment for the added zero-event subintervals is needed when a difference of sums of daily or weekly values is involved. The proper adjustment is division of each sum by the number of actual days or weeks it contains. In the above example, the number of events occurring on Sunday, Friday, and Saturday should be divided by three and the number in the second group by four. This is equivalent to creating an adjusted coefficient by replacing the zero-event subintervals with average days or average weeks and then transforming. This second procedure is identical to the procedure actually followed.

The Walsh transform applied to the robbery data is the one described by Manz [9]. The 42 coefficients that have the largest adjusted values are given in Table 1. The coefficient numbers (which start with zero) are given in both decimal and binary form. Next, the Walsh coefficients obtained with the added zero-event subintervals are given. Finally, the adjusted

Table 1. The Largest Coefficients in the Robbery Data.

Coefficient Number Decimal	Coefficient Number Binary	Walsh Coefficient	Adjusted Coefficient
0	0	2135	-
1	1	-447	-
4	100	157	178
11	1 011	173	152
128	10 000 000	-81	224
255	11 111 111	-455	-150
639	1 001 111 111	161	156
876	1 101 101 100	-163	-153
1024	10 000 000 000	729	-
1025	10 000 000 001	-169	-
1026	10 000 000 010	245	-
1152	10 010 000 000	133	237
1160	10 010 001 000	-199	-183
1280	10 100 000 000	249	145
1284	10 100 000 100	135	152
1440	10 110 100 000	-129	-140
1663	11 001 111 111	195	171
1919	11 101 111 111	127	151
1961	11 110 101 001	-139	-144
2014	11 111 011 110	143	145
2047	11 111 111 111	167	-
2048	100 000 000 000	529	-
2049	100 000 000 001	-225	-
2052	100 000 000 100	123	142
2176	100 010 000 000	193	269
2508	100 111 001 100	185	174
2626	101 001 000 010	167	168
2680	101 001 111 000	-89	-140
2687	101 001 111 111	187	141
2695	101 010 000 111	169	157
2696	101 010 001 000	153	155
2815	101 011 111 111	195	149
2852	101 100 100 100	157	145
2943	101 101 111 111	115	161
3071	101 111 111 111	319	-
3072	110 000 000 000	591	-
3073	110 000 000 001	-175	-
3074	110 000 000 010	175	-
3200	110 010 000 000	171	255
3563	110 111 101 011	-143	-145
3931	111 101 011 011	-139	-147
4095	111 111 111 111	217	-

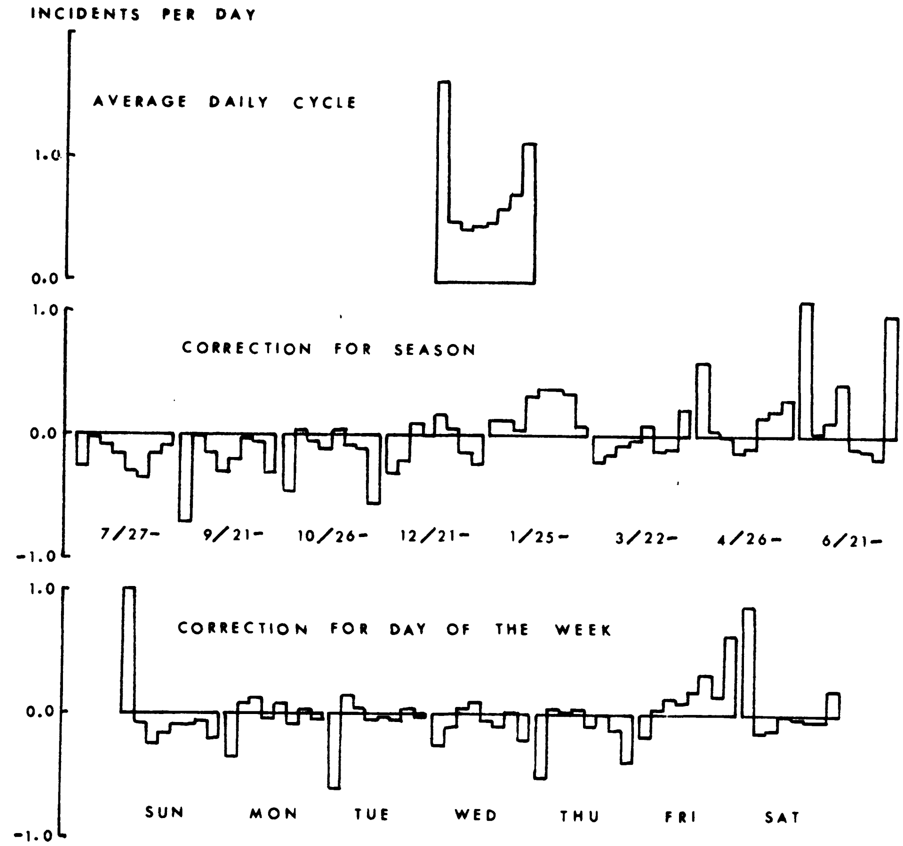

Figure 1. The Weekly and Seasonal Variation in the Daily Pattern.

coefficients are given for those Walsh coefficients that compare subintervals of unequal length. Coefficient 0 is not a difference but the total number of incidents.

Each Walsh coefficient (except 0) is the difference between two counts that can be considered independent Poisson random variables. Thus, the variance of each Walsh coefficient is estimated by the total count, 2135. Since the two counts are large enough for the normal approximation to apply, three standard deviations (which equals 139) has its usual meaning as a standard of comparison. Estimates of the variance of the adjusted coefficients vary from coefficient to coefficient. However, for the purpose of choosing the significant coefficients, the variances of the adjusted coefficients are nearly the same as those of the Walsh coefficients.

Consider the meaning of the coefficients in Table 1. Beside coefficient 0, the largest is coefficient 1024. This coefficient is the number of events that occurred in the first two and the last two subintervals of the day minus the number in the middle four subintervals. It shows that more incidents occur between 8:31 p.m. and 1:00 p.m. the next day than in the other part of the day. Since the subintervals are chosen to equalize the rate of occurrence, it follows that the reverse is true for victims in other sex-age categories. The other coefficients in Table 1 that are sums over all days of differences between parts of the day are coefficients 2047, 2048, 3071, 3072, and 4095.

The coefficients in Table 1 that show the variation of the daily pattern from quarter to quarter are coefficients 1, 1025, 1026, 2049, 3073, and 3074. Coefficient 1 compares the first half of the total interval (which ends January 24, 1970) with the second half. It shows an upward trend. The next largest in this group is coefficient 1026. It is a comparison between the middle half of the year and the first and last quarters. The quantities compared are sums over the two parts of the year of a function of the daily pattern. This function is the first two and last two subintervals of the day minus the middle four subintervals. It shows that the predominance of attacks between 8:31 p.m. and 1:00 p.m. the next day is more pronounced during daylight saving time.

The coefficients in Table 1 that show the variation of the daily pattern within the week are 128, 255, 639, 1152, 1280, 1663, 1919, 2176, 2687, 2815, 2943, and 3200. These coefficients are affected by the added zero-event subintervals. Coefficient 128 is the number of events on Sundays, Fridays, and Saturdays minus the number on Mondays, Tuesdays, Wednesdays, and Thursdays. This difference has a value of -81. Since each day of the week has an average of 305 incidents, the adjusted coefficient is 224. It shows that on a per day basis weekends have more events than weekdays.

Retransforming some of the Walsh coefficients provides an informative display of the data. The first graph in Figure 1 shows that the first and last subintervals of the day predominate. As noted above, coefficient 1024 reflects this. The second and third graphs are additive corrections to the first. The second shows the correction to the daily pattern for eight divisions of the year. As shown by coefficients 1025 and 1026, the large percentage of incidents in the first and last periods of the day is more pronounced in the last quarter of the year. The upward trend is also evident. The third graph shows the corrections by day of week. As would have been noted if coefficient 1152 had been discussed, the predominance of the first and last periods of the day is more pronounced on weekends.

Figure 1 does not display all the coefficients in Table 1. Some, such as coefficient 11, may indicate the need for finer graduation of the season. Others, such as coefficients 1284, 2626, 2680, and 2695, may indicate that the weekly pattern varies with season. The others may have a societal explanation that has not yet been uncovered. On the other hand, they may be large only by chance.

3. APPLYING THE WALSH TRANSFORM

To apply the Walsh transform, simple methods for interpreting the coefficients are needed. This section provides such methods, both an algorithm for constructing any Walsh function and the distributional properties of the coefficients. Next, application of the Walsh transform to multivariate point processes is discussed. Finally, the Walsh transform is compared, in terms of simplicity of interpretation, to the log-linear analysis of contingency tables and to spectral analysis.

For visualizing Walsh functions, defining them by specifying their sign changes is convenient. This is possible because the range of Walsh functions is ± 1. Denote the kth value of the ith Walsh function by $w(i,k)$, $0 \le i, k \le 2^M - 1$. The construction can be thought of as starting at $k = 0$ and proceeding to larger values of k. The value of $w(i,0)$ is $+1$. Let the binary representation of i be $i_{M-1}\, i_{M-2} \ldots i_0$ $(i = \sum i_p 2^p)$, and express k as $j 2^{M-1-p}$ for some odd integer j. The ith Walsh function changes sign between k-1 and k if and only if $i_p = 1$. Thus, i_0 determines whether the sign changes between the first and last half of the domain; i_1 determines whether the sign changes between the first and second quarters and between the third and fourth quarters; etc.

This definition gives the Walsh functions in an order different from the usual one [10, 11]. The ith function has i sign changes. Thus, the Walsh functions are ordered by their sequency [11], that is, by the number of sign changes they have. A fast algorithm that applies the Walsh functions in sequency order is given by Manz [9]. Note that the matrix with elements $w(i,k)$ is a symmetric Hadamard matrix. If it is multiplied by $2^{-M/2}$, it is orthogonal.

Separate sets of binary digits specify divisions within and between the periods present. For the robbery data, the first three digits $i_{11}\, i_{10}\, i_9$ specify divisions within the day; the second three digits $i_8\, i_7\, i_6$ specify divisions between days within the week; the next four digits $i_5\, i_4\, i_3\, i_2$ specify divisions between weeks within the quarter; and the last two digits $i_1\, i_0$ specify divisions between quarters. The coefficients corresponding to the margins (those that could have been obtained by superimposing the data in successive periods of one of the cycles present) have digits with the following property. The digits in the set that specifies the divisions of the period, say $i_{p+q} \ldots i_p$, are followed by digits equal to the last in the set, $i_p = i_{p-1} = \ldots$.

A 2^M-point Walsh transform can be thought of as a $2^{M'}$-point transform of a $2^{M''}$-point transform, where $M' + M'' = M$. This property is the basis for the fast algorithm. It is also useful in the interpretation of the Walsh coefficients as comparisons. Let $n(k_d,k_w,k_q,k_y)$ be the number of events in subinterval k_d of the day, day k_w of the week, week k_q of the quarter, and quarter k_y of the year. Let $w^{(M)}(p,p')$ $(0 \le p,p' \le 2^M-1)$ be the values of the 2^M-point Walsh functions. The following four steps produce Walsh coefficient i, where $i = i_d 2^9 + i_w 2^6 + i_q 2^2 + i_y$:

$$f_d(i_d,k_w,k_q,k_y) = \textstyle\sum_k w^{(3)}(i_d,k)\; n(k,k_w,k_q,k_y) \;, \tag{3.1}$$

$$f_w(i_d,i_w,k_q,k_y) = \textstyle\sum_k w^{(3)}(i_w',k)\; f_d(i_d,k,k_q,k_y),$$

$$i_w = \begin{cases} i_w' & \text{if } i_d \text{ even} \\ 7-i_w' & \text{if } i_d \text{ odd,} \end{cases} \tag{3.2}$$

$$f_q(i_d,i_w,i_q,k_y) = \textstyle\sum_k w^{(4)}(i_q',k)\; f_w(i_d,i_w,k,k_y),$$

$$i_q = \begin{cases} i_q' & \text{if } i_w \text{ even} \\ 15-i_q' & \text{if } i_w \text{ odd,} \end{cases} \tag{3.3}$$

$$f(i_d,i_w,i_q,i_y) = \textstyle\sum_k w^{(2)}(i_y',k)\; f_q(i_d,i_w,i_q,k),$$

$$i_y = \begin{cases} i_y' & \text{if } i_q \text{ even} \\ 3-i_y' & \text{if } i_q \text{ odd.} \end{cases} \tag{3.4}$$

If each of these equations were further decomposed into 2-point transforms, the result would be Manz's algorithm [9].

The adjustment for the added zero-event subintervals can now be seen. Consider (3.2) first. No adjustment is needed if f_w is the sum of f_d (the case $i_w' = 0$). Otherwise, the proper comparison is given by

$$g_w(i_d,i_w,k_q,k_y) = f_w(i_d,i_w,k_q k_y) + (w^{(3)}(i_w',1)/7)\; \textstyle\sum_k f_d(i_d,k,k_q,k_y). \tag{3.5}$$

This is equivalent to normalizing by the number of days in the subintervals being compared.

Since the adjustment for the added weeks is similar, the equations for the adjusted coefficients equivalent to (3.1) - (3.4) are easily obtained. Let

$$a(i_w') = \begin{cases} [w^{(3)}(i_w',1)]/7 & \text{if } i_w' \neq 0 \\ 0 & \text{if } i_w' = 0 \end{cases} \quad (3.6)$$

$$b(i_q') = \begin{cases} [w^{(4)}(i_q',13) + w^{(4)}(i_q',14) \\ \quad + w^{(4)}(i_q',15)]/13 & \text{if } i_q' \neq 0 \\ 0 & \text{if } i_q' = 0. \end{cases} \quad (3.7)$$

Equation (3.1) remains the same. Equations (3.2) - (3.4) become

$$g_w(i_d,i_w,k_q,k_y) = \sum_k[w^{(3)}(i_w',k) + a(i_w')]\ f_d(i_d,k,k_q,k_y), \quad (3.8)$$

$$g_q(i_d,i_w,i_q,k_y) = \sum_k[w^{(4)}(i_q',k) + b(i_q')]\ g_w(i_d,i_w,k,k_y), \quad (3.9)$$

$$g\ (i_d,i_w,i_q,i_y) = \sum_k w^{(2)}(i_y',k)g_q(i_d,i_w,i_q,k), \quad (3.10)$$

where the relation between (i_d,i_w,i_q,i_y) and (i_d,i_w',i_q',i_y') is given in (3.2) - (3.4). The adjusted coefficients can be computed from the Walsh coefficients.

In order to derive distributional properties for the Walsh coefficients, the series of events is modeled as a nonhomogeneous Poisson process. For the type of series being discussed, this assumption seems reasonable. The inhomogeneity is the most pronounced feature of such series, and the nonhomogeneous Poisson process is the simplest model with this feature. Note that adding zero-event subintervals does not make this model invalid.

The distribution theory for Walsh coefficients is simple as long as the number of coefficients considered simultaneously is small enough that the normal approximation applies. Approximate normality after transformation is a familiar property, occurring, for example, in ordinary time series analysis. Let

$$k = k_d 2^9 + k_w 2^6 + k_q 2^2 + k_y,$$

$$En(k_d,k_w,k_q,k_y) = \mu(k_d,k_w,k_q,k_y) = \mu_k,$$

$$f_i = f\ (i_d,i_w,i_q,i_y). \quad (3.11)$$

The variance of the Walsh coefficients is

$$\operatorname{var} f_i = \sum_k \mu_k \quad (3.12)$$

Since $w(i',k)\ w(i'',k) = w(i,k)$, where i is the dyadic product of i' and i" (the binary digit i_p equals $i_p' + i_p''$ (mod 2)), the covariance of coefficients i' and i" is

$$\operatorname{cov}\ (f_{i'},f_{i''}) = \sum_k w(i,k)\mu_k. \quad (3.13)$$

This can be estimated by the ith Walsh coefficient.

Consider the statistical problem of separating the adjusted coefficients that are not zero from those that are. Both because of the nature of the data and because of the adjustments for the added subintervals, conditional tests are appropriate. One approach is as follows: Tests for the coefficients that involve neither comparisons among days within a week ($i_w' = 0$) nor comparisons among weeks within a quarter ($i_q' = 0$) are conditioned only by the total number of events, $n(+,+,+,+)$. Tests for coefficients for which $i_w' = 0$ and $i_q' \neq 0$ and those for which $i_w' \neq 0$ and $i_q' = 0$ are conditioned on $n(k_d,+,+,k_y)$ $(= \sum_{j,k} n(k_d,j,k,k_y))$. Tests for the coefficients for which $i_w', \neq 0$ and $i_q' \neq 0$ are conditioned on $n(k_d,k_w,+,k_y)$. Even with the conditioning, the adjusted coefficients are approximately normal. Thus, given the variances, thresholds high enough to prevent many zero coefficients from being classified non-zero can be set using the Bonferroni inequality [6]. It can be shown that the variances of the adjusted coefficients in Table 1 are all nearly the same.

Since the Walsh transform fits an additive model to the data, extension to multivariate point processes by means of an additive model seems reasonable. However, this is not necessary. A particular Walsh coefficient can be compared among event types using a multiplicative model. Thus, the question of whether several rates of occurrence are proportional can be answered. If the number of event types is not too large, the distributional properties are simplified in the same way as in the univariate case.

For example, consider events of two types. Let the Walsh coefficients be f_i' and f_i'', respectively, and let $f_i = f_i' + f_i''$. The totals for each event type are f_0' and f_0''. The comparison appropriate to proportionality of the rates of occurrence is $f_i'/f_0' - f_i''/f_0''$. The statistic

$$c_i = \frac{f_0\ f_0'\ f_0''}{(f_0^2 - f_i^2)}\left[\frac{f_i'}{f_0'} - \frac{f_i''}{f_0''}\right]^2 \quad (3.14)$$

is the chi-square statistic usually obtained for a fourfold table. This can be seen by letting $f_i' = n_{11} - n_{12}$ and $f_i'' = n_{21} - n_{22}$, where n_{11}, n_{12}, n_{21}, n_{22} are counts. The chi-square approximation for (3.14) does not require dividing the point process into subintervals with adequately large numbers of events. Thus, as in the univariate case, the Walsh transform allows large-sample approximations to be applied. The hypothesis that overall the two rates are proportional can be tested by comparing the largest c_i with a threshold provided by the Bonferroni inequality [6].

Many authors have concluded on the basis of structural properties that multiplicative models for contingency tables are superior to additive ones [1, 5]. In particular, the fact that the hierarchy of multiplicative models contains independence and conditional independence is an important advantage. For example, it can be argued that the model fit to the robbery data in Figure 1, which is an additive composition of the daily-weekly and daily-seasonal marginals, is not as helpful for interpretation as conditional independence, the analogous multiplicative model, would have been. Further, there are problems for which multiplicative models are unavoidable because testing a particular multiplicative model is suggested a priori.

For the problem considered in this paper, the arguments for multiplicative models do not seem important enough to outweigh the advantages of the Walsh transform. In this paper, the model is a device for finding irregularities that are either so interesting or so large that they should not be distorted by the smoothing. For this purpose, the fact that the hierarchy of Walsh-transform models provides variability in the sizes of the cells is important. For example, the choice of how many divisions are needed to represent the seasonal variation in the robbery data is included in the model building.

The advantage of the Walsh transform is the simplicity of its distributional properties. Limitations on the sparseness permissible with the Pearson chi-square are reviewed by Haberman [6]. Even less is known about the distributional properties of the likelihood chi-square when the expected number in each cell is small. Sparseness is an important property of the robbery data since the 2912 cells contain 1555 zeros, 863 ones, 321 twos, 109 threes, 35 fours, 18 fives, 7 sixes, 2 sevens, 1 eight, and 1 nine.

The guidance provided by the Walsh transform is needed in the analysis of the counting function of the process that results from superimposing several days. This analysis is important when the variation during the day is large as it is for the robbery data. This counting function, which can be treated as an empirical distribution function, might show sharp changes in the rate function that can be related to other phenomena. In such an analysis, the Walsh coefficients provide guidance on what days can be superimposed without risking a severely distorted result.

Spectral analysis of point processes also fits an additive model [7]. As mentioned above, this technique is needed for periodicities with unknown frequency. However, spectral analysis is not as suitable for assessing the interactions among periodicities as the Walsh transform. With spectral analysis, interactions must be detected in the spectrum by recognizing that the frequencies of some peaks are sums or differences of the frequencies of other peaks. Further, the form of the interaction is harder to visualize with spectral analysis.

These comparisons show that the Walsh transform lies between contingency table analysis and spectral analysis, modeling the interactions as in contingency table analysis and avoiding the problems of grouping the data as in spectral analysis.

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